Iterates Properties for q-Bernstein-Stancu Operators

Yali Wang and Yingying Zhou

Abstract—In 2009, Nowak introduced q-Bernstein-Stancu polynomials $B_n^{q,\alpha}(f;x)$. When $\alpha = 0$, $B_n^{q,\alpha}(f;x)$ reduces to the well-known q-Bernstein polynomials introduced by Phillips in 1997; when $q = 1$, $B_n^{q,\alpha}(f;x)$ reduces to Bernstein-Stancu polynomials introduced by Stancu in 1968; when $q = 1$ and $\alpha = 0$, we obtain classical Bernstein polynomials. This paper deals with iterates properties of q-Bernstein-Stancu operators $B_n^q(f,q;x)$ in the case $q \in (0,1]$. $\alpha > 0, f \in \mathbb{C}[0,1]$, where both $j_\alpha \to \infty$ and $n \to \infty$.

Index Terms—q-Bernstein-Stancu polynomials, iterates properties, uniform convergence.

I. INTRODUCTION

Let $q > 0$, for each nonnegative integer $r$, we define the $q$–integer $[r]$ as

$$[r]_q = [r] = \begin{cases} 1-q^r, & q \neq 1, \\ r, & q = 1. \end{cases}$$

We then define $q$–factorial $[r]!$ as

$$[r]! = [r] = [r]_q [r-1]_q \cdots [1]_q [0]! = 1.$$

We next define a $q$–binomial coefficient as

$$\binom{n}{r}_q = \frac{[n]!}{[r]! [n-r]!}.$$

For $f \in \mathbb{C}[0,1], q > 0, \alpha \geq 0$ and each positive integer $n$, we shall investigate the following q-Bernstein-Stancu operator introduced by Nowak in 2009 \cite{1}.

$$B_n^{q,\alpha}(f;x) = \sum_{k=0}^{n} B_n^{k,\alpha}(x)f\left(\frac{k}{n}\right).$$

where

Note that empty product in (2) denotes 1.

In this case, when $\alpha = 0, B_n^{q,\alpha}(f;x)$ reduces to the well-known q-Bernstein polynomials introduced by Phillips \cite{2} in 1997:

$$B_n(q,f;x) = \sum_{k=0}^{n} B_n^{k}(x)f\left(\frac{k}{n}\right).$$

Now, we review and state some general properties of q-Bernstein-Stancu operators.

They are also degree-reducing on polynomials, that is if $P_n$ is a polynomial of degree $m$, then $B_n^{q,\alpha}(P_n)$ is a polynomial of degree $\leq \min(m,n)$.

Taking $\alpha = 0, b = 1$ in (4), we conclude that

$$\sum_{k=0}^{n} B_n^{q,\alpha}(x) = 1, \quad n \in \mathbb{N}$$

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In 2009, Nowak proved that the q-Bernstein-Stancu operators can be expressed in terms of q-differences [1]:

$$B^{q,\alpha}_n (f; x) = \sum_{k=0}^{n} \Delta_q^{k} f_0 \prod_{j=0}^{k-1} \frac{x + \alpha j}{1 + \alpha j},$$

(6)

where

$$\Delta_q^{k} f_0 = \left[ \frac{k!}{n!} \right]^q \frac{q^{k-1}}{q^k} f \left[ 0; \frac{1}{n}, \ldots, \frac{k}{n} \right].$$

Then

$$B^{q,\alpha}_n (f; x) = \sum_{i=0}^{n} \lambda_{iq,q}^{(n)} f \left[ 0; \frac{1}{n}, \ldots, \frac{k}{n} \right] \prod_{j=0}^{k-1} \frac{x + \alpha j}{1 + \alpha j},$$

(7)

where

$$\lambda_{iq,q}^{(n)} = \sum_{i=0}^{n} \left[ \frac{k!}{n!} \right]^q \frac{q^{k-1}}{q^k} f \left[ 0; \frac{1}{n}, \ldots, \frac{k}{n} \right].$$

Note that

$$\lambda_{0,q}^{(n)} = \lambda_{iq,q}^{(n)} = 1,$$

(9)

and

$$0 \leq \lambda_{i,q} \leq 1, \quad k = 0, 1, \ldots, n.$$  

(10)

at the same time, he still prove that for \( 0 < q < 1 \), \( \alpha > 0 \),

$$B^{q,\alpha}_n (1; x) = 1, \quad B^{q,\alpha}_n (t; x) = x,$$

(11)

and

$$B^{q,\alpha}_n (t^2; x) = \frac{1}{1 + \alpha} \left( x(\alpha) + \frac{x(1-x)}{n} \right).$$

(12)

He also proved some other approximation properties [1].

In recent years, q-Bernstein polynomials have been studied intensively by a number of authors. They investigated iterates properties of the Bernstein operator from a different point of view [4]–[6].

We will deal with iterates properties of q-Bernstein-Stancu operators \( B^{q,\alpha}_n (f; q, x) \) in the case \( q \in (0,1), \alpha > 0, f \in \mathbb{C}[0,1] \),

where both \( j_n \to \infty \) and \( n \to \infty \) in this paper.

It can be readily seen that for \( q \in (0,1) \), polynomials \( B^{q,\alpha}_n (q; k) \) are non negative on the interval \([0,1] \). Therefore, we get from

$$\sum_{k=0}^{n} B^{q,\alpha}_n (x) = 1 \quad \text{that} \quad \|B^{q,\alpha}_n\| = 1, \quad q \in (0,1).$$

In this paper we always assume that \( B^{q,\alpha}_n \) on \( \mathbb{C}[0,1] \) exists.

We denote the operator of linear interpolation at 0 and 1 by \( L \), i.e.,

$$L(f; x) = (1-x)f(0) + xf(1).$$

Now we give the statement of main results in this paper.

**Theorem.** For \( q \in (0,1), \alpha > 0 \), let \( \{ j_n \} \) be a sequence of positive integers such that \( j_n \to \infty \). Then for any \( f \in \mathbb{C}[0,1] \),

$$\left( B^{q,\alpha}_n \right)_{j_n} \hookrightarrow L(f; x) \quad \text{for} \quad x \in [0,1] \quad \text{as} \quad n \to \infty.$$

**II. PROOF OF THEOREM 1**

**Lemma 1.** Let \( f = r^\alpha, \ m \geq 1 \), then

$$\left( B^{q,\alpha}_n \right)(f; x) = \alpha_{\alpha} \frac{n}{m} \left[ \frac{1}{n} \right]^q \prod_{j=0}^{t-1} \left( \frac{1}{n} x + \alpha j \right) + \alpha_{\alpha} \frac{n}{m} \left[ \frac{1}{n} x + \alpha j \right] + \cdots + \alpha_{\alpha} \frac{n}{m} \left[ \frac{1}{n} x + \alpha j \right] = \min(m,n),$$

(14)

where

$$\alpha_{\alpha} \geq 0 (i = 1, \ldots, j),$$

$$\alpha_{\alpha} + \alpha_{\alpha} + \cdots + \alpha_{\alpha} = 1.$$  

Besides, for \( n \geq m \), we have

$$\alpha_{\alpha} \leq \frac{C_{m,n}}{n^m}, \quad i = 1, \ldots, m.$$  

**Proof.** It was already noticed in the introduction \( B^{q,\alpha}_n (r^\alpha; x) \) is a polynomials of degree \( \min(m,n) \). The end-point interpolation property (3) implies that for \( m \geq 1 \),

the free term of \( B^{q,\alpha}_n (r^\alpha; x) \) equals \( 0 \). Therefore, (14) is justified.

1) Representation (7) of q-Bernstein-Stancu polynomials gives the following values of coefficients in (14)

$$\alpha_{\alpha} = \frac{\lambda_{iq,q}^{(n)}}{n^m}, \quad i = 1, \ldots, m$$

(15)

where \( 0 \leq \lambda_{iq,q}^{(n)} \leq 1 \) are given by (10) Since for \( f = r^\alpha, \ f \left[ \frac{1}{n}, \ldots, \frac{1}{n} \right] \geq 0 \), the statement is proved.

2) This follows readily from the end-point interpolation properties (3) if we put \( x = 1 \) in (14).
3) Using (15) and (10) we get
\[
\alpha \leq f \left[ \frac{1}{n!}, \ldots, \frac{1}{n} \right] = f^{(k)}(x) = 0
\]
where \( \xi = \left[ \frac{1}{n} \right] \).

Hence
\[
\alpha \leq \left( \frac{m}{i} \right)^{m-1} \leq \left( \frac{m}{i} \right)^{m-1} = \frac{c_{m,i}}{n^{m-1}},
\]
as required.

4) The proof see [7].

**Lemma 2.** For all \( q \in (0,1), \alpha > 0 \), the following identity holds:
\[
B^{m,\alpha}_{s,n}(x) = \frac{\alpha}{[n]^{-1}(1 + \alpha[n-1])} \sum_{j=0}^{n-1} \left( \frac{m}{j} \right)^{m-1} \lambda^{(n)}(t;j,x) + \alpha^{(n)}(m) = \lambda^{(n)}(t;j,x).
\]

**Proof.** Let \( B^{m,\alpha}_{s,n}(x) \) be defined by (2). Then
\[
B^{m,\alpha}_{s,n}(x) = \sum_{k=1}^{n} \left( \frac{k}{n} \right)^{k} \frac{[k]^{n-1}}{[n]} B^{m,\alpha}_{s,n,k}(x) = \sum_{j=0}^{n-1} \left( \frac{m}{j} \right)^{m-1} \lambda^{(n)}(t;j,x) + \alpha^{(n)}(m) = \lambda^{(n)}(t;j,x).
\]

**Lemma 3.** For all \( q \in (0,1), \alpha > 0 \), the operator \( B^{m,\alpha}_{s,n} \) has \( n+1 \) linearly independent monic eigenvalues \( P^{(m)}(x) \), deg \( P^{(m)}(x) = m, \) \( m = 0, \ldots, n \), corresponding to the eigenvalues
\[
\lambda^{(m)}(n) = \lambda^{(n)}(m) = 1,
\]
for
\[
m = 2, \ldots, n.
\]

**Proof.** For \( m = 0,1 \), the statement is obvious due to
\[
B^{m,\alpha}_{s,n}(x + a_{m}x^{m-1} + \ldots + a_{1}x) = B^{m,\alpha}_{s,n}(x).
\]

For \( n \geq m \geq 2 \), using lemma 1, we write
\[
B^{m,\alpha}_{s,n}(x) = \frac{\lambda^{(m)}(n)}{\prod_{j=0}^{m-1} 1 + \alpha(j)} x + P^{(m)}(x)
\]
where \( P^{(m)}(x) \in \mathcal{P}_{m-1} \), and \( \lambda^{(m)}(n) \) are given by (8).

To find an eigenvector \( P^{(m)}(x) \in \mathcal{P}_{m} \) of the operator \( B^{m,\alpha}_{s,n} \), we write \( P^{(m)}(x) = x^{m} + a_{m-1}x^{m-1} + \ldots + a_{1}x \) and solve a linear system in unknowns \( a_{m-1}, \ldots, a_{1} \):
\[
B^{m,\alpha}_{s,n}(x^{m} + a_{m-1}x^{m-1} + \ldots + a_{1}x) = \frac{\lambda^{(m)}(n)}{\prod_{j=0}^{m-1} 1 + \alpha(j)} (x^{m} + a_{m-1}x^{m-1} + \ldots + a_{1}x)
\]

By lemma 1 and by letting the coefficients of \( x^{i} \) in the left equals the coefficients of \( x^{i} \) in the right, \( i = 1, \ldots, m-1 \), and arrange, we have:
\[
B^{m,\alpha}_{s,n}(x^{m}) = v_{1,1}x + v_{1,2}x^{2} + \ldots + v_{m-1,m}x^{m-1} + \frac{\lambda^{(m)}(n)}{\prod_{j=0}^{m-1} 1 + \alpha(j)} x^{m},
\]
\[
a_{m-1}B^{m,\alpha}_{s,n}(x^{m-1}) = a_{m-1} \left( v_{1,1}x + v_{1,2}x^{2} + \ldots + v_{m-2,m-1}x^{m-2} + \frac{\lambda^{(m-1)}(n)x^{m-1}}{\prod_{j=0}^{m-2} 1 + \alpha(j)} \right),
\]
\[
a_{m-2}B^{m,\alpha}_{s,n}(x^{m-2}) = a_{m-2} \left( v_{1,1}x + v_{1,2}x^{2} + \ldots + v_{m-3,m-2}x^{m-3} + \frac{\lambda^{(m-2)}(n)x^{m-2}}{\prod_{j=0}^{m-3} 1 + \alpha(j)} \right),
\]
and so on.
We get a triangular system whose determine equals

\[
\prod_{i=0}^{m-1} (1 + \alpha[i]) = 0.
\]

Then

\[
\prod_{i=0}^{m-1} (1 + \alpha[i]) = \prod_{i=0}^{m-1} (1 + \alpha[i]) = 0.
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\]

Hence there exists a unique monic polynomial of degree

\[
2 \leq m \leq n
\]

which is an eigenvector of \( B^\alpha_n \) with the eigenvalue

\[
\prod_{i=0}^{m-1} (1 + \alpha[i]) = 0.
\]

Corollary 1. For \( 2 \leq m \leq n \), and all \( q \in (0,1) \), \( \alpha > 0 \), the operator \( \prod_{i=0}^{m-1} (1 + \alpha[i]) = 0 \)

\[
\prod_{i=0}^{m-1} (1 + \alpha[i]) = 0.
\]

Lemma 4. For \( q \in (0,1) \), \( \alpha > 0 \), the following equality holds:

\[
\lim_{n\to\infty} \frac{\lambda^{(n)}}{\prod_{i=0}^{m-1} (1 + \alpha[i])} = \frac{q^{\frac{m(n-1)}{2}}}{\prod_{i=0}^{m-1} (1 + \alpha[i])}.
\]

Proof. The statement follows from Formula (17)

\[
\lim_{n\to\infty} \frac{\lambda^{(n)}}{\prod_{i=0}^{m-1} (1 + \alpha[i])} = \frac{q^{\frac{m(n-1)}{2}}}{\prod_{i=0}^{m-1} (1 + \alpha[i])}.
\]

Lemma 5. Let \( q \in (0,1) \), \( \alpha > 0 \), then for every \( m = 0, 1, \ldots \), the operator \( B^\alpha_n \) has an eigenvector \( p^i_m(x) \) which is a monic polynomial of degree \( m \), corresponding to the eigenvalue

\[
\prod_{i=0}^{m-1} (1 + \alpha[i]) = 0.
\]

Proof. Taking the limit in

\[
\prod_{i=0}^{m-1} (1 + \alpha[i]) = 0.
\]

and we note that

\[
\prod_{i=0}^{m-1} (1 + \alpha[i]) = 0.
\]

We get
Taking the limit in (16), we have

\[
\lim_{n \to \infty} \lambda_{m,q}^{(n)}(t^m, q; x) = \left[1 - \zeta(t^m) \right]^{m-1} B_{m,q}^{\alpha}(t^m, x) + \alpha \zeta(t^m) B_{m,q}^{\alpha}(t^m, x).
\]

Similarly, taking the limit in

\[
\lim_{n \to \infty} \alpha \sum_{j=0}^{m-1} \left[ q^j (n-1)^{j+1} B_{n-1}^{\alpha}(t^{j+1}, x) \right] = \alpha \sum_{j=0}^{m-1} \left[ q^j (n-1)^{j+1} B_{n-1}^{\alpha}(t^{j+1}, x) \right] \to \frac{\alpha q^j (n-1)^{j+1}}{1 - q + \alpha m - 1} - \alpha q^j (n-1)^{j+1} B_{n-1}^{\alpha}(t^{j+1}, x) \to \frac{\alpha q^j (n-1)^{j+1}}{1 - q + \alpha m - 1}.
\]

Taking the limit in (16), we have

\[
B_{m,q}^{\alpha}(t^m, x) = \lim_{n \to \infty} \lambda_{m,q}^{(n)}(t^m, x) = \left[1 - \zeta(t^m) \right]^{m-1} B_{m,q}^{\alpha}(t^m, x) + \alpha \zeta(t^m) B_{m,q}^{\alpha}(t^m, x).
\]

The coefficient \( \lambda_{m,q}^{(n)} \) of \( x^m \) in \( B_{m,q}^{\alpha}(t^m, x) \) equals

\[
\lambda_{m,q}^{(n)} \equiv \frac{\alpha n^{-1} (1 - q) \lambda_{m,q}^{(n-1)} + \alpha q \sum_{j=0}^{m-1} \left[ (q - 1)^{j+1} B_{n-1}^{\alpha}(t^{j+1}, x) \right]}{1 - q + \alpha n^{-1} \sum_{j=0}^{m-1} \left[ (q - 1)^{j+1} B_{n-1}^{\alpha}(t^{j+1}, x) \right] + \alpha n^{-1} \sum_{j=0}^{m-1} \left[ (q - 1)^{j+1} B_{n-1}^{\alpha}(t^{j+1}, x) \right].
\]

This means

\[
\lim_{n \to \infty} \lambda_{m,q}^{(n)} = \left[1 - \zeta(t^m) \right]^{m-1} B_{m,q}^{\alpha}(t^m, x) + \alpha \zeta(t^m) B_{m,q}^{\alpha}(t^m, x).
\]

Recursively

\[
\lambda_{m,q}^{(n)} \equiv \frac{\alpha n^{-1} (1 - q) \lambda_{m,q}^{(n-1)} + \alpha q \sum_{j=0}^{m-1} \left[ (q - 1)^{j+1} B_{n-1}^{\alpha}(t^{j+1}, x) \right]}{1 - q + \alpha n^{-1} \sum_{j=0}^{m-1} \left[ (q - 1)^{j+1} B_{n-1}^{\alpha}(t^{j+1}, x) \right] + \alpha n^{-1} \sum_{j=0}^{m-1} \left[ (q - 1)^{j+1} B_{n-1}^{\alpha}(t^{j+1}, x) \right].
\]
Then,
\[
\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right) y_{m-1}^{(n)} = \left(\Lambda_{m,q}^{(n)}\right)^{j} p_{m-1}^{(n)} - \left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} p_{m-1}^{(n)}.
\]
It follows from (13) and (18) that \(\|p_{m-1}^{(n)}\| \leq 2\). Since \(\Lambda_{m,q}^{(n)} \rightarrow 0\) as \(n \rightarrow \infty\), we have
\[
\left(\Lambda_{m,q}^{(n)}\right)^{j} p_{m-1}^{(n)} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]
it can be readily seen from (18) and Lemma 4 that
\[
p_{m-1}^{(n)}(x) = Q_{m-1}^{(n)}(x) + \delta_{n}(x),
\]
i.e.
\[
p_{m-1}^{(n)}(x) = Q_{m-1}^{(n)}(x) + \delta_{n}(x),
\]
where \(Q_{m-1} \in \mathcal{P}_{m-1}\), and \(\delta_{n}(x) \rightarrow 0\) as \(n \rightarrow \infty\).

Thus, \(\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} p_{m-1}^{(n)} = \left(\Lambda_{m,q}^{(n)}\right)^{j} p_{m-1}^{(n)} + \left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} \left(\delta_{n}\right)\),

where \(\left\|\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} \left(\delta_{n}\right)\right\| \leq \left\|\delta_{n}\right\|\), because of (13). This means that \(\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} \left(\delta_{n}\right) \rightarrow \infty\) as \(n \rightarrow \infty\).

By the induction assumption
\[
\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} Q_{m-1}^{(n)}(x) = \alpha x + d \in \mathcal{P}_{j},
\]
Therefore, \(\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} p_{m-1}^{(n)} = \alpha x + d \quad \text{as} \quad n \rightarrow \infty\),
or
\[
\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right) y_{m-1}^{(n)} = \alpha x + d + \omega_{n}(x),
\]
where \(\omega_{n} \rightarrow 0\) as \(n \rightarrow \infty\).

By corollary 1, the operator \(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\) are invertible on \(\mathcal{P}_{m-1}\) for \(n \geq m\) and
\[
\lim_{n \rightarrow \infty}\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} = \frac{1}{q^{(m-1)}} \prod_{j=0}^{m-1} (1 + \alpha(j)),
\]
where by corollary 2, \(A_{n,q}\) is also invertible on \(\mathcal{P}_{m-1}\). Hence
\[
\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} \rightarrow A_{n,q}^{-1} \quad \text{as} \quad n \rightarrow \infty,
\]
and it follows that
\[
\left\|\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j}\right\| \leq M \quad \text{for some} \quad M > 0.
\]

Therefore,
\[
y_{m-1}^{(n)} = \left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} (cx + d) + \left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} \left(\omega_{n}\right).
\]
Since \(\left\|\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} \left(\omega_{n}\right)\right\| \leq M \left\|\omega_{n}\right\| \rightarrow 0\) as \(n \rightarrow \infty\) and \(\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} \rightarrow A_{n,q}^{-1}\) as \(n \rightarrow \infty\), we conclude that
\[
y_{m-1}^{(n)} \rightarrow A_{n,q}^{-1} (cx + d) = ax + b \in \mathcal{P}_{j}.
\]
Thus, \(B_{n}^{q,a}(x) = ax + b\).

The induction is completed and it follows that for any polynomial \(\mathcal{P}\),
\[
\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} p(x) = L(p;x) \quad \text{for} \quad x \in [0,1], \quad \text{as} \quad n \rightarrow \infty.
\]

II. Let \(f \in \mathbb{C}[0,1]\), and let \(\varepsilon > 0\) be given. Then \(f(x) = p(x) + \delta(x)\), where \(p \in \mathcal{P}\), and \(\delta(x) \leq \varepsilon\). We have
\[
\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} f = \left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} p + \left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} \left(\delta\right).
\]
Since \(\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} \left(\delta\right) \leq L(p)\), there exists \(n_{0} \in \mathbb{N}\) such that
\[
\left\|\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} \left(\delta\right)\right\| < \varepsilon \quad \text{for all} \quad n > n_{0}.
\]

Obviously, \(\left\|\delta\right\| < \varepsilon\), and finelly we obtain
\[
\left\|\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} f - L(f)\right\| \leq \left\|\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} p - L(p)\right\| + \left\|\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} \left(\delta\right)\right\| < 3\varepsilon, \quad \text{for all} \quad n > n_{0}.
\]
Thus,
\[
\left\|\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} f - L(f)\right\| \leq \left\|\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} p - L(p)\right\| + \left\|\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} \left(\delta\right)\right\| < 3\varepsilon, \quad \text{for all} \quad n > n_{0}.
\]

Thus, \(\left(\Lambda_{m,q}^{(n)} I - B_{n}^{q,a}\right)^{j} f(x) \leq L(f;x)\), for \(x \in [0,1]\) as \(n \rightarrow \infty\).
III. CONCLUSION

In this paper, iterates properties for q-Bernstein-Stancu operators are studied, the result of iterates properties for q-Bernstein-Stancu operators is obtained. This study is just a small step in this area. To make further progress in this direction, one could try to study other properties for q-Bernstein-Stancu operators, such as shape-preserving and convergence properties to make this area perfected and enriched.

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