A Note on Two Point Taylor Expansion III

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***Abstract*—If a function is analytic on an interval, then the function is expressed as the Taylor expansion about a point in the interval. Furthermore, possibility of Taylor expansions of functions about two or three points has also been studying as useful expressions in several fields of mathematical sciences. In this paper, we show the following main result by estimating values of divided differences: Let f be a piecewise polynomial continuous function such that f is a polynomial p on the interval $[1/3, \infty)$ and f is a polynomial q on the interval $(-\infty, 1/3]$. Then, we show that f is expressed as the two point Taylor expansion about $-1, 1$ with the multiplicity weight $(2, 1)$ on the interval (α, β), where α is the solution of $(x+1)^2(x-1) = -32/27$ with $\alpha < -1$ and β is the solution of $(x+1)^2(x-1) = 32/27$ with $\beta > 1$.**

***Index Terms*—Polynomial interpolation, Hermite interpolation, Taylor expansion, two point taylor expansion.**

I. INTRODUCTION

As is well known, polynomial approximation has a long history and has established the foundation of approximation theory. In particular, interpolations by polynomials play a very important part of polynomial approximation. Before stating the purpose of this note, we briefly review Hermite interpolation by polynomials.

Let A be an infinite subset of the real line R and let f be a real-valued function on A. For any given $(n+1)$ distinct points $X: x_0, \cdots, x_n$ in the interior of A and for any sequence of positive integers $k_0, \cdots, k_n$, if f is sufficiently differentiable at $x_0, \cdots, x_n$, then there exists a unique approximating polynomial $p_{f,X(k_0,\cdots,k_n)}(x)$ to f which is of degree at most $m(= k_0 + \cdots + k_n - 1)$ such that

$$p^{(j)}_{fX(k_0,\cdots,k_n)}(x_i) = f^{(j)}(x_i),\ 0 \le i \le n, 0 \le j \le k_i - 1 \quad (1)$$

The points $x_0, \cdots, x_n$ and the polynomial $p_{f,X(k_0,\cdots,k_n)}$ are called *nodes* and the *Hermite interpolating polynomial to f at $x_0, \cdots, x_n$ with multiplicities* $k_0, \cdots, k_n$, respectively. We well know that for one node $X: x_0$ with multiplicity n, the Hermite interpolating polynomial $p_{f,\ X(n)}$ to f is the Taylor polynomial of f about x_0, that is,

$$p_{f,X(n)}(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x - x_0)^{n-1}. \quad (2)$$

Furthermore, if f is infinitely differentiable at x_0 and if there exists a positive number ρ such that

$$f(x) = \lim_{n\to\infty} p_{f,X(n)}(x) \quad \text{for all } x \in (x_0 - \rho, x_0 + \rho), \quad (3)$$

Then f has the Taylor expansion of f about x_0 on $(x_0 - \rho, x_0 + \rho)$. Hence, we make the following definition.

Definition 1. Let f be a real-valued function on a subset A of the real line R. If there exist a list X consisting of m distinct nodes $x_0, \cdots, x_{m-1}$ in the interior of A and positive integers $w_0, \cdots, w_{m-1}$ such that f is infinitely differentiable at $x_0, \cdots, x_{m-1}$ and then it is said that f has the m *point Taylor expansion about* $x_0, \cdots, x_{m-1}$ *with the multiplicity weight* $(w_0, \cdots, w_{m-1})$ *on* A.

$$\lim_{n\to\infty} p_{f,X(w_0 n,\cdots,w_{m-1}n)}(x) = f(x) \quad \text{for all } x \in A, \quad (4)$$

The notion of two point or m point Taylor expansion is not new. One can see some representations of $p_{f,X(n,\cdots,n)}(x)$ in Davis [1] and the theory of m point Taylor expansion in the complex plane in Walsh [2]. López and Temme [3], [4] stated how m point Taylor expansion in the complex plane can be used in deriving uniform asymptotic expansions of contour integrals of the form $I(\lambda;\alpha) = \int_C g(z)e^{-\lambda f(z,\alpha)}dz$, where α is a vector of parameters and the phase function $f(z,\alpha)$ has finite saddle points. Our first contact with two point Taylor expansion starts with Runge example: Let $f(x) = \frac{1}{1+25x^2}$, $x \in [-1,1]$. Let $X_n : x_0^{(n)} = -1$, $x_1^{(n)} = -1 + \frac{1}{n}, \cdots, x_n^{(n)} = 0, \cdots, x_{2n}^{(n)} = 1, n \in \boldsymbol{N}$ be the system of equally spaced nodes and let $P_{f,X_n}(x)$, $n \in \boldsymbol{N}$ be the polynomials of degree at most 2n which interpolates the function f at nodes in X_n. Then, it is well known that $\overline{\lim}_{n\to\infty}\|f - P_{f,X_n}\|_\infty = +\infty$, where $\|\cdot\|_\infty$ denotes the supremum norm on $C[-1,1]$. On the other hand, if we take the list of two nodes $X : -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$, then we already know that $\lim_{n\to\infty}\|f - P_{f,X(n,n)}\|_\infty = 0$. Because, for an analytic function $g(x)$ on $[-1,1]$ and for a list of nodes $X : x_0, x_1, \cdots, x_n$ in $[-1,1]$ it holds that for each $x \in [-1,1]$,

$$g(x) - P_{g,X}(x) = \frac{1}{2\pi i}\int_C \frac{(x-x_0)\cdots(x-x_n)}{(z-x)(z-x_0)\cdots(z-x_n)} f(z)dz, \quad (5)$$

where C is a simple, closed, rectifiable curve of $\mathbf{C}$ whose interior C^i contains $[-1,1]$ and g is regular on $C \cup C^i$ (see p.165 in Mori [5]). Indeed, if we take the list of nodes X: $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ and consider $P_{f,X(n,n)}$ for $f(x) = \frac{1}{1+25x^2}, x \in [-1,1]$ and if we take a simple, closed, rectifiable curve C such that C^i contains $[-1,1]$ and f is regular on $C \cup C^i$, then from (5), it holds that

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$$f(x) - P_{f,X(n,n)}(x) = \frac{1}{2\pi i}\int_C \frac{\left(x^2-\frac{1}{2}\right)^n}{(z-x)\left(z^2-\frac{1}{2}\right)^n} f(z)dz,$$
$$x \in [-1,1].$$

Hence, if we consider a simple, closed, rectifiable curve C such that $\left|z^2-\frac{1}{2}\right| > \frac{1}{2}, z \in C$ and the poles $\pm\frac{i}{5}$ of $\frac{1}{1+25z^2}$ are not contained in C^i, since we see that

$$\left|\frac{x^2-\frac{1}{2}}{z^2-\frac{1}{2}}\right| < 1,\ \ x \in [-1,1],\ z \in C,$$

We obtain $f(x) - P_{f,X(n,n)}(x) \to 0\ (n \to \infty),\ x \in [-1,1]$.

Hence, we are much interested in a problem: what functions on $[-1,1]$ have m point Taylor expansions. Kitahara, Chiyonobu and Tsukamoto [6] and Kitahara, Yamada and Fujiwara [7] studied two point Taylor expansion of spline functions on R with one knot, which are not always continuous at the knot. In [6], the following result was shown:

Theorem 1. Let f be a function on R, which is expressed as

$$f(x) = \begin{cases} p(x) & x \in [0,\infty) \\ q(x) & x \in (-\infty,0) \end{cases}, \tag{6}$$

where p and q are polynomials of degree at most n. Let $P_\ell, \ell \in \boldsymbol{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities ℓ, ℓ. Then, the following assertions hold:

f has the two point Taylor expansion about $-1, 1$ with multiplicity weight $(1, 1)$ on $\left(-\sqrt{2},0\right) \cup \left(0,\sqrt{2}\right)$, that is,

$$\lim_{\ell\to\infty} P_\ell(x) = f(x) \quad \text{for all } x \in \left(-\sqrt{2},0\right) \cup \left(0,\sqrt{2}\right) \tag{7}$$

Moreover, if $p(0) = q(-0)$, then f has the two point Taylor expansion about $-1, 1$ with multiplicity weight $(1, 1)$ on $\left(-\sqrt{2},\sqrt{2}\right)$, that is,

$$\lim_{\ell\to\infty} P_\ell(x) = f(x) \quad \text{for all } x \in \left(-\sqrt{2},\sqrt{2}\right). \tag{8}$$

In this note, we will show the following result of two point Taylor expansions which is related to Theorem 1.

Theorem 2. Let f be a function on R, which is expressed as

$$f(x) = \begin{cases} p(x) & x \in \left[\frac{1}{3},\infty\right) \\ q(x) & x \in \left(-\infty,\frac{1}{3}\right) \end{cases} \tag{9}$$

where p and q are polynomials of degree at most n. Let $Q_\ell, \ell \in \boldsymbol{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities $2\ell, \ell$. Let α be the real number with $\alpha < -1$ and $(\alpha+1)^2(\alpha-1) = -32/27$ and β the real number with $\beta > 1$ and $(\beta+1)^2(\beta-1) = 32/27$. Then, the following assertions hold:

f has the two point Taylor expansion about $-1, 1$ with the multiplicity weight $(2, 1)$ on $(\alpha, 1/3) \cup (1/3, \beta)$ that is,

$$\lim_{\ell\to\infty} Q_\ell(x) = f(x) \quad \text{for all } x \in \left(\alpha,\frac{1}{3}\right) \cup \left(\frac{1}{3},\beta\right). \tag{10}$$

Moreover, if $p(1/3) = q(1/3-0)$, then f has the two point Taylor expansion about $-1, 1$ with the multiplicity weight $(2, 1)$ on (α, β), that is,

$$\lim_{\ell\to\infty} Q_\ell(x) = f(x) \quad \text{for all } x \in (\alpha,\beta). \tag{11}$$

II. Preliminaries

We review some well known results and a definition which are related to Hermite interpolating polynomials.

Proposition 3. (see p. 365 in Kincaid and Cheney [8]) Let $x_0 \le x_1 \le \cdots \le x_n$ be a list of nodes. In the list of nodes, only distinct nodes $z_0, \cdots, z_r$ appear and each node $z_i, i = 0, \cdots, r$ is just appeared k_i times. Let f be sufficiently differentiable at $z_0, \cdots, z_r$. Then, there exists a unique polynomial p of degree at most n satisfying that

$$p^{(j)}(z_i) = f^{(j)}(z_i), \qquad i = 0, \cdots, r, j = 0, \cdots, k_i - 1 \tag{12}$$

In Proposition 3, each positive integer $k_i, i = 0, \cdots, r$ is called the multiplicity at x_i. Divided differences of functions can be defined from this proposition.

Definition 2. Let $x_0 \le x_1 \le \cdots \le x_n$ be a list of nodes and let f be sufficiently differentiable at $x_0, \cdots, x_n$. Then we call the coefficient of x^n of the polynomial p with the property (12) stated above *the* n-th order divided difference of f at $x_0, \cdots, x_n$ and denote its n-th order divided difference by $f[x_0, \cdots, x_n]$.

By Definition 2, it is easily seen that the divided difference $f[x_0]$ of a function f at a point x_0 is equal to $f(x_0)$. We have the recursive formula to calculate divided differences of functions.

Proposition 4. (see p. 372 in Kincaid and Cheney [8]) *Let* $x_0 \le x_1 \le \cdots \le x_n$ be a list of nodes and let f be sufficiently differentiable at $x_0, \cdots, x_n$. Then the divided differences obey this recursive formula:

$$f[x_0, \cdots, x_n] = \begin{cases} \dfrac{f[x_1, \cdots, x_n] - f[x_0, \cdots, x_{n-1}]}{x_n - x_0} & (x_0 \ne x_n) \\ \dfrac{f^{(n)}(x_0)}{n!} & (x_0 = x_n) \end{cases} \tag{13}$$

If data points $(x_i, f(x_i)), i = 0, \cdots, n$ are given, then we can construct the following divided difference table $T_f[x_0, \cdots, x_n]$ from them. By Proposition 4, the $(i+1)$-th order divided differences in the table are calculated from the i-th order divided differences.

x_0	$f[x_0]$			
		$f[x_0, x_1]$		
x_1	$f[x_1]$			
		$f[x_1, x_2]$		
x_2	$f[x_2]$			
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$f[x_0, \cdots, x_n]$
x_{n-2}	$f[x_{n-2}]$			
		$f[x_{n-2}, x_{n-1}]$		
x_{n-1}	$f[x_{n-1}]$			
		$f[x_{n-1}, x_n]$		
x_n	$f[x_n]$			

Divided Difference Table $T_f[x_0, \cdots, x_n]$

In the divided difference table stated above, we call the column vector consisting of the i-th order divided differences *the i-th order column* for convenience.

Notation. Let $x_0 \le x_1 \le \cdots \le x_n$ be a list of nodes and let f be sufficiently differentiable at $x_0, \cdots, x_n$. In the list of nodes, only distinct points $z_0, \cdots, z_r$ appear and each point $z_i, i = 0, \cdots, r$ is just appeared k_i times. To make sure of multiplicities, we express for the divided difference $f[x_0, \cdots, x_n]$. And the divided difference table $T_f[x_0, \cdots, x_n]$ is also denoted by $T_f[z_0, \cdots, z_r; k_0, \cdots, k_r]$.

$$f[z_0, \cdots, z_r; k_0, \cdots, k_r]$$

The following proposition is a basic statement, but it is a key result to prove our theorems.

Proposition 5. *Let* $(a \le) x_0 \le x_1 \le \cdots \le x_n (\le b)$ be a list of nodes and let f be a real-valued function on an interval $[a, b]$ which is sufficiently differentiable at $x_0, \cdots, x_n$. If p is the Hermite interpolating polynomial to f at $x_0, \cdots, x_n$, then for each $x \in [a, b]$

$$f(x) - p(x) = f[x, x_0, \cdots, x_n](x - x_0)(x - x_1) \cdots (x - x_n) \quad (14)$$

III. Proof of Theorem 2

Now we are in position to prove Theorem 2.

Proof. Let $Q_\ell, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities $2\ell, \ell$. For any given $t \in (\alpha, \beta)$, from Proposition 5 we have

$$|f(t) - Q_\ell(t)| = \left|f[-1, t, 1; 2\ell, 1, \ell](t + 1)^{2\ell}(t - 1)^\ell\right| = |f[-1, t, 1; 2\ell, 1, \ell]| \cdot |(t + 1)^2(t - 1)|^\ell . \quad (15)$$

Since $\sup_{t \in (\alpha,\beta)} |(t + 1)^2(t - 1)| = \frac{32}{27}$, it is sufficient to show that

$$\lim_{\ell \to \infty} |f[-1, t, 1; 2\ell, 1, \ell]| \left(\frac{32}{27}\right)^\ell = 0 \quad (16)$$

We prove (16) for the case $t \in \left(\alpha, \frac{1}{3}\right)$. Because, the other cases $t = \frac{1}{3}$ (if f is continuous at $= \frac{1}{3}$) or $t \in \left(\frac{1}{3}, \beta\right)$, we can show (16) in an analogous way to the case $t \in \left(\alpha, \frac{1}{3}\right)$.

Now we give an estimation $|f[-1, t, 1; 2\ell, 1, \ell]|$ for $t \in \left(\alpha, \frac{1}{3}\right)$ and much larger ℓ than n. Let us see the part of $T_f[-1, t, 1; 2\ell, 1, \ell]$ from the 0-th column to the $(n + 1)$-th column.

nodes	the 0-th column		the $(n+1)$-th column
-1	$q(-1)$		$f[-1; n+2]$
$\vdots$	$\vdots$		$\vdots$
$\vdots$	$\vdots$		$f[-1; n+2]$
-1	$q(-1)$		$f[-1, t; n+1, 1]$
t	$q(t)$	$\cdots$	$\vdots$
1	$p(1)$		$f[t, 1; 1, n+1]$
$\vdots$	$\vdots$		$f[1; n+2]$
$\vdots$	$\vdots$		$\vdots$
1	$p(1)$		$f[1, n+2]$

Since $p(x)$ and $q(x)$ are polynomials of degree at most n, we easily have

$$f[-1; n + 2] = \frac{q^{(n+1)}(t)}{(n + 1)!} = 0 \quad (17)$$

$$f[1; n + 2] = \frac{p^{(n+1)}(t)}{(n + 1)!} = 0 \quad (18)$$

Furthermore, we have

$$f[-1, t; n + 1, 1] = \frac{1}{(t + 1)^{n+1}} \left\{ q(t) - \sum_{i=0}^{n} \frac{q^{(i)}(-1)}{i!} (t + 1)^i \right\} \quad (19)$$

$$f[t, 1; 1, n + 1] = \frac{1}{(t - 1)^{n+1}} \left\{ q(t) - \sum_{i=0}^{n} \frac{p^{(i)}(1)}{i!} (t - 1)^i \right\} \quad (20)$$

Nothing that $p(x)$ and $q(x)$ are polynomials of degree at most n, we get

$$f[-1, t; n + 1, 1] = \frac{q(t) - q(t)}{(t + 1)^{n+1}} = 0 \quad (21)$$

And

$$f[t, 1; 1, n + 1] = \frac{q(t) - p(t)}{(t - 1)^{n+1}} . \quad (22)$$

By (17), (18) and (21), the part of $T_f[-1, t, 1; 2\ell, 1, \ell]$ from the 0-th column to the $(n + 1)$-th column is as follows:

nodes	the 0-th column		the $(n+1)$-th column
-1	$q(-1)$		0
$\vdots$	$\vdots$		$\vdots$
$\vdots$	$\vdots$		0
-1	$q(-1)$		$f[-1, t, 1; n, 1, 1]$
t	$q(t)$	$\cdots$	$\vdots$
1	$p(1)$		$f[t, 1; 1, n+1]$
$\vdots$	$\vdots$		0
$\vdots$	$\vdots$		$\vdots$
1	$p(1)$		0

In order to give an estimation of $|f[-1, t, 1; 2\ell, 1, \ell]|$, we put two positive numbers

$$M \coloneqq \max_{i=0,\cdots,n} |f[-1, t, 1; n - i, 1, i + 1]| \; r \coloneqq \frac{1}{1 - t}.$$

We write $\boldsymbol{a} \le \boldsymbol{b}$ for any column vector $\boldsymbol{a} = (a_i)_{1 \le i \le s}, \boldsymbol{b} = (b_i)_{1 \le i \le s} \in \boldsymbol{R}^s$ such that $a_i \le b_i, i = 1, \cdots, s$. For any column vector $\boldsymbol{a} \in \boldsymbol{R}^s$, let $\boldsymbol{a}(k)$ be the column vector in $\boldsymbol{R}^s$ such that the k-th element is equal to a_k and the other elements are equal to 0, and put $\widetilde{\boldsymbol{a}} = (|a_i|)_{1 \le i \le s}$.

Let $\boldsymbol{a}_k = \left(a_i^{(k)}\right)_{1 \le i \le 3\ell - k + 1}, n + 1 \le k \le 3\ell$, be the k-th column of $T_f[-1, t, 1; 2\ell, 1, \ell]$ and we define the column vectors $\boldsymbol{b}_k = \left(b_i^{(k)}\right)_{1 \le i \le 3\ell - k + 1}, n + 1 \le k \le 3\ell$ as follows:

First, we set the column vector $\boldsymbol{b}_{n+1} = \left(b_i^{(n+1)}\right)_{1 \le i \le 3\ell - n} \in \boldsymbol{R}^{3\ell - n}$ such that

$$b_i^{(n+1)} = \begin{cases} 0 & 1 \le i \le 2\ell - n, 2\ell + 2 \le i \le 3\ell - n \\ M & otherwise. \end{cases} \quad (23)$$

For each $k, n + 2 \le k \le \ell$, we put the column vector $\boldsymbol{b}_k = \left(b_i^{(k)}\right)_{1 \le i \le 3\ell - k + 1} \in \boldsymbol{R}^{3\ell - k + 1}$ satisfying that

$$b_i^{(k)} = \begin{cases} r^{k-(n+1)} \cdot M & i = 2\ell + 1 \\ \dfrac{b_{i+1}^{(k-1)} + b_i^{(k-1)}}{2} & otherwise. \end{cases} \quad (24)$$

Furthermore, we set the column vectors $\boldsymbol{b}_k = \left(b_i^{(k)}\right)_{1 \le i \le 3\ell - k + 1} \in \boldsymbol{R}^{3\ell-k+1}, \ell + 1 \le k \le 3\ell$ such that

$$b_i^{(k)} = \frac{b_{i+1}^{(k-1)} + b_i^{(k-1)}}{2}, \quad 1 \le i \le 3\ell - k + 1\,. \quad (25)$$

Then, we easily see that $\tilde{\boldsymbol{a}}_{n+1} \le \boldsymbol{b}_{n+1}$. And we observe that $\tilde{\boldsymbol{a}}_k \le \boldsymbol{b}_k, n + 2 \le k \le 3\ell$. Because, for each $k, n + 2 \le k \le \ell$, $\boldsymbol{a}_k = \left(a_i^{(k)}\right)_{1 \le i \le 3\ell - k + 1} \in \boldsymbol{R}^{3\ell-k+1}$ is obtained by

$$a_i^{(k)} = \begin{cases} \{(-1) \cdot r\}^{k-(n+1)} f[t, 1; 1, n + 1] & i = 2\ell + 1 \\ \dfrac{a_{i+1}^{(k-1)} - a_i^{(k-1)}}{2} & otherwise \end{cases} \quad (26)$$

And for each $k, \ell + 1 \le k \le 3$, $\boldsymbol{a}_k = \left(a_i^{(k)}\right)_{1 \le i \le 3\ell - k + 1} \in \boldsymbol{R}^{3\ell-k+1}$ is got from

$$a_i^{(k)} = \frac{a_{i+1}^{(k-1)} - a_i^{(k-1)}}{2}, \qquad 1 \le i \le 3\ell - k + 1. \quad (27)$$

In particular, we have $\tilde{\boldsymbol{a}}_{3\ell} = |f[-1, t, 1; 2\ell, 1, \ell]| \le \boldsymbol{b}_{3\ell}$.

To evaluate $\boldsymbol{b}_{3\ell}$, let w_m be a linear map from $\boldsymbol{R}^m (m \ge 2)$ to $\mathbf{R}^{m-1}$ such that for all $\boldsymbol{c} = (c_i)_{1 \le i \le m} \in \boldsymbol{R}^m$

$$\left(w_m(\boldsymbol{c})\right)_i = \frac{c_i + c_{i+1}}{2}, \qquad i = 1, \cdots, m - 1\,. \quad (28)$$

And let us consider the real-valued function φ on $\boldsymbol{R}^m$ which is defined by

$$\varphi(\boldsymbol{c}) = w_2 \circ w_3 \circ \cdots \circ w_m(\boldsymbol{c}) \text{ for all } \boldsymbol{c} \in R^m. \quad (29)$$

If $\boldsymbol{e}_m^{(k)} = (a_i) \in \boldsymbol{R}^k$ denotes the column vector such that $a_m = 1$ and $a_i = 0$ otherwise, from the definition of $\boldsymbol{b}_{3\ell}$ and the linearity of φ, we have

$$\begin{aligned} \boldsymbol{b}_{3\ell} &= \varphi(\boldsymbol{b}_{n+1}) + \varphi\left(\boldsymbol{b}_{n+2}(2\ell + 1)\right) + \cdots + \varphi\left(\boldsymbol{b}_\ell(2\ell + 1)\right) \\ &= \varphi\left(\boldsymbol{b}_{n+1}(2\ell - n + 1) + \cdots + \boldsymbol{b}_{n+1}(2\ell + 1)\right) \\ &\quad + \varphi\left(\boldsymbol{b}_{n+2}(2\ell + 1)\right) + \cdots + \varphi\left(\boldsymbol{b}_\ell(2\ell + 1)\right) \\ &= M\left\{\varphi\left(\boldsymbol{e}_{2\ell-n+1}^{(3\ell-n)}\right) + \varphi\left(\boldsymbol{e}_{2\ell-n+2}^{(3\ell-n)}\right) + \cdots + \varphi\left(\boldsymbol{e}_{2\ell+1}^{(3\ell-n)}\right)\right\} \\ &\quad + rM\varphi\left(\boldsymbol{e}_{2\ell+1}^{(3\ell-n-1)}\right) + r^2M\varphi\left(\boldsymbol{e}_{2\ell+1}^{(3\ell-n-2)}\right) + \cdots \\ &\quad + r^{\ell-n-1}M\varphi\left(\boldsymbol{e}_{2\ell+1}^{(2\ell+1)}\right) \\ &= A + B, \end{aligned} \quad (30)$$

where

$$\begin{aligned} A &= M\left\{\varphi\left(\boldsymbol{e}_{2\ell-n+1}^{(3\ell-n)}\right) + \varphi\left(\boldsymbol{e}_{2\ell-n+2}^{(3\ell-n)}\right) + \cdots + \varphi\left(\boldsymbol{e}_{2\ell+1}^{(3\ell-n)}\right)\right\} B \\ &= rM\varphi\left(\boldsymbol{e}_{2\ell+1}^{(3\ell-n-1)}\right) + r^2M\varphi\left(\boldsymbol{e}_{2\ell+1}^{(3\ell-n-2)}\right) \\ &\quad + \cdots \\ &\quad + r^{\ell-n-1}M\varphi\left(\boldsymbol{e}_{2\ell+1}^{(2\ell+1)}\right). \end{aligned} \quad (31)$$

For any nonnegative integer m, k with $m \ge k$, $\binom{m}{k}$ denotes the binomial coefficient, that is,

$$\binom{m}{k} = \frac{m!}{k!\,(m - k)!}.$$

By using Pascal's triangle, since it hold that for all positive integers s, t with $s \ge t$, we have

$$\varphi\left(\boldsymbol{e}_t^{(s)}\right) = \frac{1}{2^{s-1}}\binom{s - 1}{s - t} \quad (32)$$

$$\begin{aligned} A &= \frac{2^{n+1} \cdot M}{2^{3\ell}} \sum_{i=0}^{n} \binom{3\ell - (n + 1)}{\ell - (i + 1)} \\ &< \frac{2^{n+1} \cdot M}{2^{3\ell}} \cdot (n + 1)\binom{3\ell}{\ell} \end{aligned} \quad (33)$$

And

$$\begin{aligned} B &= \frac{2^{n+1} \cdot M}{2^{3\ell}} \sum_{s=1}^{\ell-(n+1)} (2r)^s \cdot \binom{3\ell - (s + n + 1)}{\ell - (s + n + 1)} \\ &= \frac{2^{n+2} \cdot M}{2^{3\ell}} r\left\{\binom{3\ell - (n + 2)}{\ell - (n + 2)} + 2r\binom{3\ell - (n + 3)}{\ell - (n + 3)} + \cdots \right. \\ &\quad \left. + (2r)^{\ell-(n+2)}\binom{2\ell}{0}\right\} \\ &= \frac{2^{n+2} \cdot M}{2^{3\ell}} r\binom{3\ell - (n + 2)}{\ell - (n + 2)}\left\{1 + 2r\frac{\ell - (n + 2)}{3\ell - (n + 2)} \right. \\ &\quad + 4r^2\frac{\ell - (n + 2)}{3\ell - (n + 2)} \cdot \frac{\ell - (n + 3)}{3\ell - (n + 3)} + \cdots \\ &\quad \left. + (2r)^{\ell-(n+2)}\frac{\ell - (n + 2)}{3\ell - (n + 2)} \cdots \frac{1}{2\ell + 1}\right\}. \end{aligned} \quad (34)$$

Since

$$(0 <)\frac{1}{1 - \alpha} < r < \frac{1}{1 - \frac{1}{3}} = \frac{3}{2} \quad (35)$$

And

$$\frac{1}{3} > \frac{\ell - (n + 2)}{3\ell - (n + 2)} > \frac{\ell - (n + 3)}{3\ell - (n + 3)} > \cdots > \frac{1}{2\ell + 1}, \quad (36)$$

Putting $R = 2r \cdot \frac{\ell-(n+2)}{3\ell-(n+2)}$, we obtain $(0 <)R < 2 \cdot \frac{3}{2} \cdot \frac{1}{3} = 1$.

Hence we have the following estimation of B:

$$\begin{aligned} B &< \frac{2^{n+2} \cdot M}{2^{3\ell}} r\binom{3\ell - (n + 2)}{\ell - (n + 2)}\left\{1 + R + R^2 + \cdots + R^{\ell-(n+2)}\right\} \\ &= \frac{2^{n+2} \cdot M}{2^{3\ell}} r\binom{3\ell - (n + 2)}{\ell - (n + 2)}\frac{1 - R^{\ell-(n+1)}}{1 - R} \\ &< \frac{2^{n+2} \cdot rM}{1 - R} \cdot \frac{1}{2^{3\ell}}\binom{3\ell}{\ell} \end{aligned} \quad (37)$$

By (33) and (37), we see that

$$\begin{aligned} \boldsymbol{b}_{3\ell} &= A + B \\ &< \left\{(n + 1)2^{n+1} \cdot M + \frac{2^{n+2} \cdot rM}{1 - R}\right\} \cdot \frac{1}{2^{3\ell}}\binom{3\ell}{\ell} \\ &= \frac{\lambda}{2^{3\ell}}\binom{3\ell}{\ell}, \end{aligned} \quad (38)$$

where $\lambda = (n + 1)2^{n+1} \cdot M + \frac{2^{n+2} \cdot rM}{1-R}$. Since $|(t + 1)^2(t -$

$1)| \leq \frac{32}{27}$, from (38), we obtain

$$|f(t)-Q_\ell(t)| = |f[-1,t,1;2\ell,1,\ell]|\cdot|(t+1)^2(t-1)|^\ell \leq \frac{\lambda}{2^{3\ell}}\binom{3\ell}{\ell}\cdot\left(\frac{32}{27}\right)^\ell = \lambda\left(\frac{4}{27}\right)^\ell\binom{3\ell}{\ell}. \quad (39)$$

By use of Starling formula, for any positive number ρ_1, ρ_2 with $\rho_1 < 1 < \rho_2$, there exists a positive integer N satisfying that

$$\rho_1\sqrt{2\pi k}\left(\frac{k}{e}\right)^k < k! < \rho_2\sqrt{2\pi k}\left(\frac{k}{e}\right)^k \quad (40)$$

For all $k \geq N$. Hence, for any positive integer $\ell \geq N$, it holds that

$$\left(\frac{4}{27}\right)^\ell\binom{3\ell}{\ell} = \left(\frac{4}{27}\right)^\ell\frac{(3\ell)!}{\ell!\,(2\ell)!} < \left(\frac{4}{27}\right)^\ell\frac{\rho_2\sqrt{2\pi\cdot3\ell}\left(\frac{3\ell}{e}\right)^{3\ell}}{\rho_1\sqrt{2\pi\cdot\ell}\left(\frac{\ell}{e}\right)^\ell\cdot\rho_1\sqrt{2\pi\cdot2\ell}\left(\frac{2\ell}{e}\right)^{2\ell}} = \left(\frac{4}{27}\right)^\ell\cdot\frac{\rho_2}{\rho_1{}^2}\sqrt{\frac{3}{4\pi\ell}}\left(\frac{27}{4}\right)^\ell = \frac{\rho_2}{\rho_1{}^2}\sqrt{\frac{3}{4\pi\ell}} \quad (41)$$

This means that $Q_\ell(t)$ converges to $f(t)$ as ℓ tends to infinity.

In the case $t \in \left(\frac{1}{3},\beta\right)$, we have an estimation

$$\boldsymbol{b}_{3\ell} < \left\{(n+1)2^{n+1}\cdot M + \frac{2^{n+2}\cdot rM}{1-R}\right\}\cdot\frac{1}{2^{3\ell}}\binom{3\ell}{\ell}, \quad (42)$$

where

$$M := \max_{i=0,\cdots,n}|f[-1,t,1;(n+1)-i,1,i]|$$
$$r := \frac{1}{t+1}$$
$$R := 2r\cdot\frac{2\ell-(n+2)}{3\ell-(n+2)}.$$

In the case $t = \frac{1}{3}$, Since $p\left(\frac{1}{3}\right) = q\left(\frac{1}{3}\right)$, we have an estimation,

$$\boldsymbol{b}_{3\ell} < n2^{n+1}M\cdot\frac{1}{2^{3\ell}}\binom{3\ell}{\ell}. \quad (43)$$

This completes the proof.

IV. Conclusions and Future Work

Theorem 1 states that spline functions on $(-\sqrt{2},\sqrt{2})$ with one knot 0 have the two point Taylor expansions about $-1, 1$ with the multiplicity weight $(1,1)$. Moreover, in this note, we prove that spline functions on (α,β) with one knot $\frac{1}{3}$ possess the two point Taylor expansions about $-1, 1$ with the multiplicity weight $(2,1)$. From these conclusions, we observe that the position of one knot stated in Theorem 1 and 2 is closely related to the multiplicity weight. Hence, we give problems which lead to a next step.

A. Problems

1) Find spline functions with one knot which have two point Taylor expansions about $-1, 1$ with the multiplicity weight (m,n), where m, n are positive integers.
2) Let us consider spline functions with one knot which have two point Taylor expansions about $-1, 1$ with the multiplicity weight (m,n). Then find relations between the multiplicity weight and positions of one knot.

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