On the Exact Ordering Operators and Their Applications

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***Abstract*—Exact and general exact ordering methods are reviewed. Firstly, the exact ordering method is introduced, and few theorems were given to assign the conditions needed to locate the position of a required object among a group of objects to be ordered in a certain manner in three classes .Secondly, the exact ordering method is generalized to any odd number of classes (m). In both cases and if the required object class is put in the middle of other classes then the required object is located exactly as the object in the middle of all objects provided that we arrange the objects orderly in three groups in the first case and in m groups in the second generalized one and where certain defined steps are to be followed; in general m steps are required to determine the required object exactly and where m is the odd number of classes. The possibility of making the subject more interesting, deeper and handled in a sophisticated manner, through the introduction of exact ordering operators, is then discussed; this is by no means complete and this matter will constitute the subject of a future work .Finally few different applications are suggested in physics ,in operational research, in sorting files and in postal mailing. Its use as a practical demonstration with playing cards is also mentioned.**

***Index Terms*—Class, exact, ordering, required object.**

I. Introduction

Ordering things or arranging objects in a particular manner is a very important problem in many fields [1]-[4]. In this paper, we will deal with an ordering method which gives a procedure to determine exactly the location of a specific object out of n objects provided that we know the class to which the object went to in each step of the procedure. By that we mean that if n is divisible by s (s is odd and represents the constant number of classes formed in each step); then the only information we may have about the object is that it has gone to the *i* th class where $1 \le i \le s$. Such situations may occur in practice as will discuss later.

First, we will introduce the case of s being equal to 3, then we generalize to any odd s, and from there we try to proceed further and suggest exact ordering operators. Finally, we give an insight to the practical aspect of the problem [5], [6].

A. Some Considerations

Consider that we arrange the n objects in s classes in order in such a manner that filling the classes takes place gradually and orderly so that the (s+1) th object goes to the first class and the (s+2) th object to the second class and so on (the operation is classification mod s). After all n objects are arranged, the class containing the required object is put in the central position of the classes, i.e. its position is the ($\frac{s+1}{2}$) th irrespective of the order of the other classes.

If we take these considerations into account we can then proceed with various cases.

Case 1: In this case we take $n = 3^l$, then we have the following theorem

Theorem 1: If $n = 3^l$ objects are arranged in three classes in an orderly fashion and the class containing the particular object is put as the second one in each step, then l steps are needed to determine the position of the r.o. and its position r is given by $r = \frac{(3^l + 1)}{2}$.

Proof: After the completion of the first arrangement and putting the class containing the r.o. as the second class (step 1) and if r is the order of the object, then

$$3^{l-1} + 1 \le r \le 2 \times 3^{l-1} \quad (1)$$

After Step 2 we have

$$3^{l-1} + 3^{l-2} + 1 \le r \le 2 \times 3^{l-1} - 3^{l-2} \quad (2)$$

After Step I, we get

$$\begin{aligned} &3^{l-1} + 3^{l-2} + \ldots\ldots + 3^{l-i} + 1 \le r \\ &\le 2 \times 3^{l-1} - 3^{l-2} - \ldots - 3^{l-i} \end{aligned} \quad (3)$$

After l steps we obtain

$$\begin{aligned} &3^{l-1} + 3^{l-2} + \ldots\ldots + 3^{l-i} + \ldots.. + 3^0 + 1 \le r \le \\ &2 \times 3^{l-1} - 3^{l-2} - \ldots - 3^{l-i} - \ldots.. - 3^0 \end{aligned} \quad (4)$$

But we can easily see from the last inequality that

$$\begin{aligned} r &= 3^{l-1} + 3^{l-2} + \ldots\ldots + 3^{l-i} + \ldots.. + 3^0 + 1 \\ &= 2 \times 3^{l-1} - 3^{l-2} - \ldots - 3^{l-i} - \ldots.. - 3^0 \\ &= \frac{3^l + 1}{2} \end{aligned} \quad (5)$$

Two trivial examples which satisfy the previous theorem are when we have $n = 3^0$ (and $n = 3^1$) objects; a less trivial one is when the number of objects is $n = 3^2$.

Case 2: Here, we still consider 3 classes but with $n = 3m_1$ where m_1 is not necessarily divisible by 3; i.e. $m_1 = 3m_2 + k$ where $k = 1\ or\ 2$, and we have the more general theorem for the case of 3 classes.

Theorem 2: If $n = 3m_1$ and $3^{l-1} < n < 3^l$ are objects to be arranged in three classes ordered with the required-object

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class always put as the second one in each step of the arrangements, then l steps are needed to determine the object order exactly. Moreover, its position is given by $r = \frac{(n+1)}{2}$ if n is odd and $r = n/2$ *or* $r = n/2+1$ if n is even.

Proof: Since $n = 3m_1$; and $3^{l-1} < n < 3^l$, therefore after step 1, the r.o. will be squeezed to the center within a width of objects equal to m_1.

Now if $m_i = 3m_{i+1} + k$; with $3^{l-i-2} < n < 3^{l-i-1}$ and $k = 1$ *or* 2, then after step i the r.o. gets squeezed to the center within a maximum width of $m_i + 1$.

After $(l-1)$ steps, the maximum width of squeezing will be $m_{l-1} + 1$; *and* $1 < m_{l-1} < 3$. Hence, after l step, the r.o. has to be at the center [5].

B. Important Remarks

1) Note that the two theorems 1 and 2 can be combined into one general theorem and the first is treated as a special case of the general one .However, it is worthwhile to show them independently so as to get a better feeling of the problem and moreover to check different approaches .Note also that for even n there are two centers.
2) We can always make n divisible by 3, by addition or subtraction of few units or objects.
3) We expect that, instead of 3 classes, the two theorems are valid for any odd-numbered classes (i.e. 5, 7, 9, 11,….).

C. Two Illustrative Examples

In this subsection, we give two illustrative examples to verify theorems 1 and 2.

Example 1: If we take $n = 3^2$ as the number of objects and if the r.o. is denoted by β and assume after finishing step1 we have the arrangement:

cde / β ab / fgh

Then, we arrange the ordering as described above to get

c β f /dag / ebh

To finish step 2 we put the r.o.class in the middle to get

dag / c β f / ebh

From which we see that the r.o. is at position 5 as expected from theorem 1. (note that the number of classes here is 3)

Example 2: Take $n = 15 = 3 \times 5$ as the number of objects and let the r.o. be denoted by λ, then suppose that after step 1 we have the following arrangement

ABCDE / FGHI λ / JKLMN

Arranging the ordering, we have

ADG λ L / BEHJM / CFIKN

Putting the r.o. class in the middle we get (step 2)

BEHJM / ADG λ L / CFIKN

Again arranging the ordering, we obtain

BJDLI / EMGCK / HA λ FN

Now with the completion of step 3, we have

BJDLI / HA λ FN / EMGCK

Hence we see that the r.o λ is at position 8 as expected from theorem 2.

II. Generalization

To proceed with generalization we give the following few definitions:

A. Definitions

Definition 1: The required object class (roc) is the class containing the object to be located.

Definition 2: A step is the completion of the process of filling the classes orderly and putting the roc in the middle of the other classes irrespective of their orders.

Definition 3: A width of a class of objects (or elements) is the number of elements in that class.

Now, with the above definitions we proceed to present the following theorems:

Theorem 3: If $n = m^l$ are objects to be arranged orderly in (odd) m classes, then l steps are required to locate any required object and its position is given by $r = (n+1)/2$.

Proof: It clear that after step *i* the the r.o. gets squeezed to the center within a width equal to m^{l-i} and surely after l steps, the r.o. will be exactly at the center.

Theorem 4: If $n = m \times s$,where $m^{l-1} < n \le m^l$ and s is not necessarily divisible by m (m is odd), are objects to be arranged orderly in m classes, then l steps are required to determine the position of any r.o. and its order r is given by $r = (n+1)/2$ if n is odd, and $r = n/2$ *or* $r = n/2+2$ if n is even.

Proof: The proof is a direct consequence of the previous theorem if one observes that the maximum width for the roc in this case is equal to m^{l-i} after step *i*.

B. More Illustrative Examples

Again, we give two more illustrative examples to fulfill theorems 3 and 4.

Example 3: Let $n = 25 = 5^2$, then according to theorem 3, $m = 5$ and $l = 2$; therefore the r.o. is obtained in two steps and its position is the 13 th position.

We work this example in details, where we assume that the r.o. is denoted by Σ , and after step 1 we have the ordering

ABCDE / FGHIJ / Σ LMNO / PQRST / UVWXY

Now we arrange orderly to get

AF Σ PU / BGLQV / CHMRW / DINSX /EJOTY

Putting the roc in the middle of all classes, we get (Step 2)

BGLQV / CHMRW / AF Σ PU / DINSX / EJOTY

From the last step we see that the position of the object (Σ) is the 13th as expected from theorem 3 [6].

Example 4: Let $n = 15 = 5 \times 3$ then since $5 < 15 < 5^2$ and according to theorem 4 the required object which we will assign as the letter A will be determined in two steps at position 8(note that number of classes here will be 5).This is done as follows;

Assume that the r.o. A, after step 1, appears in the arrangement as

bcd/efg/Ahi/jkl/mno

Now if we arrange the objects orderly we get

bgk/cAl/dhm/ein/fjo

With the completion of step 2 we have

fjo/ein/cAl/bgk/dhm

Hence we see that the required object A is at position 8 as expected from theorem 4.

III. Exact Ordering Operators (Suggested Work)

In order to proceed on exact ordering, I believe that it should be formulated in operator theory; hence we start with the following definitions and notations:

Let P be a set of $n = m^l$ elements with m being odd, then we have the following definitions

1) An ordering $O_m(P)$ of P is a permutation of its elements $\{a,b,c,........\}$ such that they satisfy $a \equiv b \pmod m$. The residues of the classes are $e = 1,2,3,.....m(\equiv 0)$.
2) The required element class (rec) is the class containing the required element (re) to be located exactly.
3) A set of elements in a class is a set of elements of the ordering $O_m(P)$ in that class.
4) The length L of a set of elements is the number of elements in that set; so L for any class is given by $L = m^{l-1}$.
5) The order of an element in a class is its position in that class.
6) The center of a class is that element whose order in that class is $(\frac{m^{l-1}+1}{2})$.
7) A central ordering $O_{mc}(P)$ of the ordering $O_m(P)$ is permutation of P such that the residue of rec is $e = (m+1)/2$; irrespective of the orders of the other classes .So residues are changeable.
8) A step operator S is defined by the successive application of $O_m(P)$ and $O_{mc}(P)$ once, i.e. $S = O_{mc}(P)\, O_m(P)$.
9) An exact ordering $O_E(P)$ is defined as $O_E(P) = S^l(P)$.

Now with these notations and definitions one can proceed with formulating a solid theory of exact ordering operators, and this will constitute the subject of a future study.

However, we may proceed a little further and give the following important theorem:

Theorem 5: If P is a set of $n = m^l$ elements, then $O_E(P)$ will determine a required element (RE) exactly and its order will be given by $(\frac{n+1}{2})$.

Proof: It is clear that, in general, applying S^j will result in squeezing the RE to the center of class $(\frac{m+1}{2})$ within a length of $L_s = m^{l-j}$ .Hence, applying $O_E(=S^l)$ implies that $L_s = m^{l-l} = 1$; and this means that the RE is at the center.

Note that the theorem can be generalized to cases of n elements with n divisible by m and such that $m^{l-1} < n < m^l$ .In this case O_E needs to be applied l times to get the RE at the center .The proof is similar to the above one.

Example 5: Let $P=\{A,B,C,D,E,F,G,H,\lambda\}$; here we see that $n = 9 = 3^2$ and let λ be the RE to be located exactly.

To see the effect of applying exact ordering operators, let us arrange *P* as shown in Table I below:

TABLE I: First Arrangement of *P*.

1	2	3	4	5	6	7	8	9
A	B	C	D	E	F	G	H	λ

Now, Operating with O_3 we get the ordering as in Table II

TABLE II: The Arrangement after Applying O_3.

1	2	3	4	5	6	7	8	9
A	D	G	B	E	H	C	F	λ

To complete an S operation, apply O_{3c} to get the arrangement in Table III.

TABLE III: The Arrangement After an *S* Operation.

1	2	3	4	5	6	7	8	9
A	D	G	C	F	λ	B	E	H

Apply again O_3 to obtain the ordering as in Table IV below

TABLE IV: The Arrangement After another O_3 Operation.

1	2	3	4	5	6	7	8	9
A	C	B	D	F	E	G	λ	H

TABLE V: The Ordering in its Final Stage-Completing S^2.

1	2	3	4	5	6	7	8	9
A	C	B	G	λ	H	D	F	E

And with an application of O_{3c} we have the final arrangement as in Table V

Therefore, we have operated with S^2 on the completion of this step; and we got the RE λ at the center ($\frac{3^2+1}{2}$) as expected from theorem 5.

It is being noted that many applications of exact ordering may arise and this will be discussed in the conclusion part [7], [8].

IV. Conclusion

An Exact ordering method was introduced where a required object is to be located, exactly, among n objects arranged in 3 classes. If $n = 3^l$ is the number of objects and certain defined steps were followed, and then l steps are needed to locate the required object exactly. A similar procedure is adapted to generalize the method to any odd-numbered classes and very similar results were obtained for determining the required object exactly .Exact ordering operators were then elaborated on and few definitions were given with the aim of getting a deeper insight to the subject. A modest effort was presented to illustrate a theorem and a sample example to show the use of exact ordering operators and how to apply them .However, the work is far from complete and the subject will be a matter of a future study.

It is to be noted here that exact ordering may be of use in many fields; for instance it may prove useful in computers in sorting files. It is clear also that it might be useful in combinatorics and in group theory.

We may consider the implications of exact ordering in physics in designing new detectors which can be more efficient .Threshold detectors, in which materials are used that require neutrons above certain energy to cause activations ,are an example. The method can also be useful in different branches of statistical mechanics.

Moreover, we may think of its applications in postal mailing and in criminal investigations .Another probable application is the field of operations research in queuing theory.

Finally, the given illustrative examples can be demonstrated as a trick with playing cards and with the distributor blind-folded and a partner as a monitor.

REFERENCES

[1] H. G. Cambell and R. E. Spencer, *Finite Mathematics*, 1st ed., N.Y. USA, Collier Macnillan,1974, ch. 4, pp. 136-163.

[2] N. Jacobson, *Lectures in Abstract Algebra*, vol. III, 1st ed., N.Y., USA, American Book Company, 1964, ch. 5, pp. 236-239.

[3] F. S. Hillier and G. J. Lieberman, *Introduction to Operations Research*, 8th ed., New Delhi, India, McGraw-Hill Company, 2008, ch. 18, pp. 871-872.

[4] N. W. Roueche and V. H. Graves, *Business Mathematics*, 8th ed., Upper Saddle River, N.J., USA, Prentice Hall, 2001, ch. 4, pp. 61-63.

[5] A. M. Awin, "An exact ordering method," *Int. J. Math. Edu. Sci. Technol*., vol. 15, no. 4, pp. 537-539, July-August 1984.

[6] A. M. Awin, "A general exact ordering method," presented at the Australian Mathematical Society 34th Annual Meeting, Townsville July 2-7, 1990.

[7] M. Bielewicz, S. Kilim, E. Strugalska-Gola, M. Szuta, and A. Wojciechowski, "Yttrium as a new threshold detector for fast neutron energy spectrum (>10 MeV) measurement," *Journal of the Korean Physical Society*, vol. 59, no. 2, pp. 2014-2017, August 2011.

[8] N. H. Shah, R. M. Gor, and H. Soni, *Operations Research*, New Delhi, India, Prentice Hall, 2007, ch. 12, pp. 335-371.

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