A Comparative Study between Mixed and Displacement Model and the Effect of the Numerical Integration in the Axisymmetric Shell Finite Element Analysis

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***Abstract*—In this paper the influence of the choice of the model in the formulation of the axisymmetric shell finite element type is presented. A comparison between results obtained by the use of two elements, CAXI_L and CAXI_K is given. The first element is based on standard mixed formulation Hellinger - Reissner and the second one is formulated with displacement model. The variation of the results according to the number of the Gauss integration points is discussed. A Matlab program is elaborated to calculate the displacements for each element.**

***Index Terms*—Axisymmetric behavior, formulation model, matlab programming, numerical integration.**

I. Introduction

According to shell theory, we can distinguish two types of finite elements:

Finite elements where the transverse shearing is taken into account (Rissner-Mindlin theory).

Finite elements where the effect of the transverse shearing is not taken into account (Love-Kirchoff theory).

Several elements were developed since 1960, the first one formulated (1963) in the field of this type of shell structures is a truncated element which is suitable for revolution shells and is based on Love-Kirchoff theory ([1]-[4]). Currently, the most used element for the analysis of this type of structures is CAXI_K element [5], for this type, the field of displacement U is linear and W is cubic. With regard to the elements based on the Rissner-Mindlin, CAXI_L element [5] was proposed and tested. A simple and powerful element based on the displacement model was formulated in [6], the components of U and β are linear and W quadratic, and the integration is done with three Gauss points for the membrane, two for the bending and one for the transverse shearing.

The objective of this work is to highlight the influence of the choice of model in the formulation of the finite elements of axisymmetric shell type, thus the influence of the variation of the number of the Gauss points for the numerical integration type; for this, we carried out the development of two programs called Axisym CAXI_L and CAXI_K with Matlab. The first program treats the CAXI_L element, which is based on Rissner-Mindlin theory, and second is related to the CAXI_K element, which is based on Love-Kirchoff theory.

Manuscript received May 10, 2013; revised June 28, 2013. A Comparative study between mixed and displacement model and the effect of the numerical integration in the axisymmetric shell finite elements analysis.

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II. Axi-Symetric Shells Thoery

A. Love-Kirchoff Theory

The following assumptions [7] have to be considered: Geometrical assumption of linearization: Displacements and deformations remain small.

Assumption of material linearization: The material obeys the Hook's law

The transverse normal stress is neglected $\sigma_z = 0$.

The cross-sections, normal in the medium plan not deformed, remain plane and perpendicular to the medium plan deformed $\gamma_{\alpha z} = 0$, $\gamma_{\beta z} = 0$ and $\varepsilon_z = 0$

Displacement model:

The relations efforts resulting-strains are given by:

$$[N] = \lfloor H_m \rfloor \{e\} + \lfloor H_{mf} \rfloor \{\chi\} \quad (1)$$

$$[M] = \lfloor H_{mf} \rfloor \{e\} + \lfloor H_f \rfloor \{\chi\} \quad (2)$$

N: efforts resulting from membrane

M: efforts resulting from bending (moments)

With $\langle e \rangle = \langle\ e_s \quad e_\theta\ \rangle \quad \langle \chi \rangle = \langle\ \chi_s \quad \chi_\theta\ \rangle \quad (3)$

e_s, e_θ : Membrane strains according to S (meridian) and θ (circumferential)

χ_s, χ_θ : Curvatures according to s and θ.

The displacement model corresponds to (principle of virtual work):

$$W = W_{\text{int}} - W_{ext} = 0 \quad (4)$$

$$W_{\text{int}} = 2\pi \int_s \left(\langle e^* \rangle \left(\lfloor H_m \rfloor \{e\} + \lfloor H_{mf} \rfloor \{\chi\} \right) + \langle \chi^* \rangle \left(\lfloor H_{mf} \rfloor \{e\} + \lfloor H_f \rfloor \{\chi\} \right) \right) r ds \quad (5)$$

e^*, χ^* : Membrane strains and virtual curvatures respectively.

B. Rissner Mindlin Theory

Geometrical assumption of linearization: Displacements is strains remain small.

Assumption of material linearization: The material obeys the law of Hooke

The transverse normal constraint is negligible: $\sigma_z = 0$

DOI: 10.7763/IJMO.2013.V3.298

Mixed models in transverse shearing

$$W = W_{\text{int}} - W_{ext} = 0$$

$$W_{\text{int}} = 2\pi \int_s \begin{matrix} \langle e^* \rangle \left([H_m]\{e\} + [H_{mf}]\{\chi\} \right) + \\ \langle \chi^* \rangle \left([H_{mf}][e] + [H_f]\{\chi\} \right) + \gamma^* T_S + T_S^* \left(\gamma - H_c^{-1} T_S \right) \end{matrix} \, rds \quad (6)$$

Ts, *T*s*: real and virtual shearing action followings. Hc : Shearing stiffness.

III. ELEMENTS FORMULATION

A. CAXI_L Element

The finite element CAXI_L [5] is a truncated element with two nodes as presented in Fig.1, its formulation is based on the theory of Reissner-Mindlin. The model used for this element is the mixed model in transverse shearing. We suppose that the shell is discretized by a succession of truncated cones defined by the end nodes on the meridian curve.

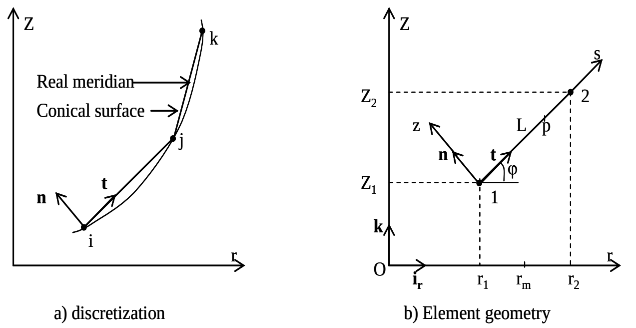

Fig. 1. Truncated element CAXI_L (geometry)

The approximations of the displacement field of U, W and of β are linear in s and L and the shearing action Ts is constant

$$U = N_1 U_1 + N_2 U_2 \quad W = N_1 W_1 + N_2 W_2 \quad (7)$$

$$\beta = N_1 \beta_1 + N_2 \beta_2$$

$$\text{with: } N_1 = 1 - s/L \quad ; \quad N_2 = s/L \quad (8)$$

The Strains are:

Deformations of membrane e_s, e_θ

The curvatures are χs, $\chi\theta$.

The transverse shearing is γ

The element stiffness matrix can be evaluated numerically with the reduced integration method of internal work W^e_{int}

$$W^e_{\text{int}} = \langle u_n^* \rangle [K] \{u_n\} \quad (9)$$

$$\text{With: } [K] = [K_{mf}] + [K_c] \quad (10)$$

$$[k_{mf}] = 2\pi \left([B_m]^T \left([H_m][B_m] + [H_{mf}][B_f] \right) + [B_f]^T \left([H_{mf}][B_m] + [H_f][B_f] \right) r_m L \right) \quad (11)$$

$$[k_c] = \{k_{m/T}\} \frac{1}{K_T} \langle k_{m/T} \rangle = 2\pi \{B_c\} H_c r_m L \langle B_c \rangle \quad (12)$$

with:

$[k_c]$: transverse shearing stiffness matrix.

$[k_{mf}]$: membrane bending stiffness matrix for an isotropic material

$$H_c = k.G.h \qquad G = E/2(1+\nu) \quad (13)$$

$K = \frac{5}{6}$ (Transverse shearing correction factor)

$$[H_m] = \frac{\text{Eh}}{(1-\nu^2)} \begin{vmatrix} 1 & \nu \\ \nu & 1 \end{vmatrix} \quad [H_f] = \frac{\text{Eh}^3}{12(1-\nu^2)} \begin{vmatrix} 1 & \nu \\ \nu & 1 \end{vmatrix} \quad (14)$$

$$[H_{mf}] = \frac{\text{Eh}^3 S}{12(1-\nu^2)\text{r}_\text{m}} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \quad (15)$$

Resulting efforts (normal effort and bending moment) can be evaluated with (1) and (2)

B. CAXI_K Element

This finite element is a truncated in shape as shown in Fig. 2. Its formulation based on the Kirchhoff theory [5] and the displacement model. The curvilinear components U (s) and W (s) are defined by linear approximations and cubic of hermitian type respectively. The numerical integration used is of Gauss type with two points for the evaluation of the stiffness matrix $[k^e]$.

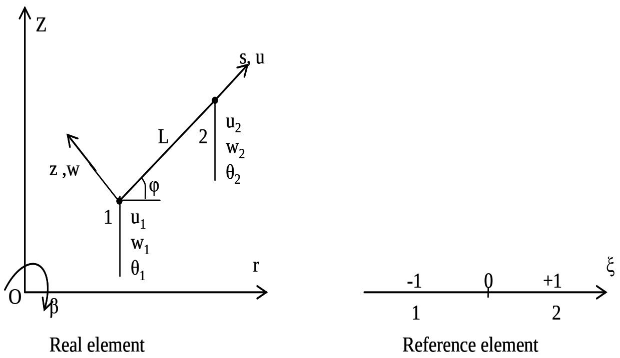

Fig. 2. Truncated element CAXI_K

$$[k]_{loc} = 2\pi \int_{s0}^{L} [k_\xi] ds = 2\pi \int_{-1}^{1} [k_\xi] \frac{L}{2} d\xi \quad (16)$$

with:

$$[k_\xi] = \left([B_m]^T \left([H_m][B_m] + [H_{mf}][B_f] \right) + [B_f]^T \left([H_{mf}][B_m] + [H_f][B_f] \right) \right) \quad (17)$$

The numerical integration according to the Gauss method is:

$$[k]_{loc} = 2\pi \sum_{i=1}^{2} [k_\xi (\xi = \xi_i)] \omega_i \frac{L}{2} \quad (18)$$

with

$$\xi_i = \pm 1/\sqrt{3} \text{ and } \omega_i = 1 \quad (19)$$

After the evaluation of $[k]_{loc}$, and before the assembling of the matrices; it is necessary to transform the variables $\{u_n\}_{loc}$ defined in the local coordinate of the element according to the nodal variables of the cylindrical reference. The transformation matrix $[T]$ is given by:

$$[T]=\begin{vmatrix}[t] & [0]\\ 0 & 1\end{vmatrix} \qquad [t]=\begin{vmatrix}[t]^T & 0\\ [0] & [t]\end{vmatrix}$$

$$[Q]=[\ t \quad n\ ]=\begin{vmatrix}C & S\\ S & C\end{vmatrix} \tag{20}$$

Thus we can write: $\left[k^e\right]=[T]^T[k]_{loc}[T]$. (21)

IV. Numerical Application

In this section, the presented elements are applied to the analysis of a cylindrical shell as indicate in Fig. 3 fixed at one end is free from the other end. Uniformly distributed load is applied at the free end. The results obtained by the Axisym program for both elements (CAXI_K and CAXI_L), and those of software ANSYS as presented in Fig. 5 are compared with the analytical solution given in [8].

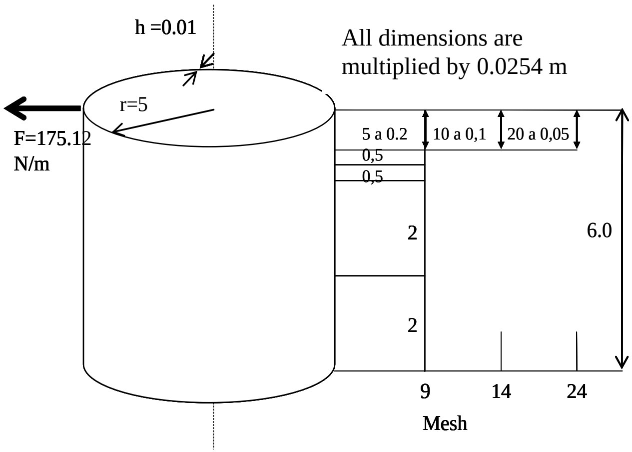

Fig. 3. Cylinder loaded at the free end.

TABLE I: Radial Displacement (m) at Free Loaded End (x0.0254/1000)

Mesh	ANSYS	SOL Ref [8]	Prog: CAXI_K	Error	Prog: CAXI_L	Error
9	2,876	2,874	2,815	2,05%	2,875	0,03%
14	2,876		2,861	0,45%	2,875	0,03%
24	2,876		2,871	0,10%	2,875	0,03%

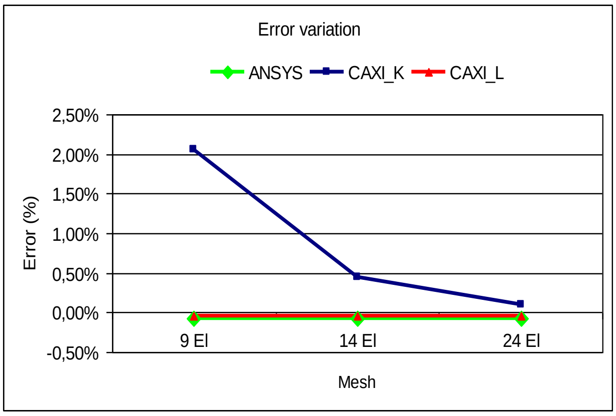

Fig. 4. Convergence of the displacement at the free end

TABLE II: Displacement Results According to Number on Integration Points (x 0.0254/1000)

Mesh	SOL [8]	Prog: CAXI_K 1 pt integration	Prog: CAXI_K 3 pts integration	Prog: CAXI_K 4 pts integration	Prog: CAXI_K 2 pts integration
24 elements	2,874	1514	2,871	2,871	0,41

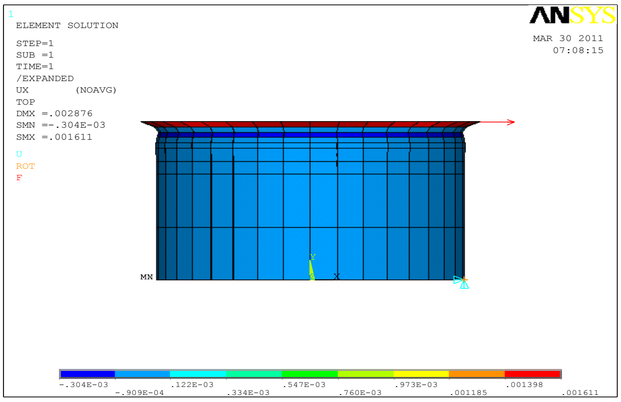

Fig. 5. Deformed structure (ANSYS)

Comments:

According to the results presented in Table I and Fig.4 for the variation of the error compared to the analytical solution, it can be notice that CAXI_L element gives better results than CAXI_K element.

In Table II we observe that the change of the number of integration points for CAXI_K element does not carry any change on the solution for the displacement even we use a more than 2 points of integration. For the case of one integration point is used, the result will be unacceptable.

For the second element CAXI_L, if we use 2 or more points, the result will be obtained with a great error, therefore only the reduced integration will be applicable.

V. Conclusion

The choice of the mathematical model leads to very important results in the applications studied; we notice the advantage of the CAXI_L element, which is based on a mixed formulation compared to CAXI_K element based on displacement model formulation, even where for the case where the analytical solution is based on Love-Kirchoff theory, therefore we can say that if the mixed variational formulation is used (standard Hellinger - Reissner), we can develop another mixed model which could give more precision.

For the shell elements, the increases in the numbers of integration points does not carry any improvement in the solution, contrary it gives for some cases bad results, which is noticed in the application for the element CAXI_L. for this case only the integration reduced will be gives good performances. Also if we use a numerical diagram of integration with more than two Gauss points for CAXI_K element, the results will be identical to those given with two points. We should mention that, using only one Gauss point for this type of elements gives unacceptable results.

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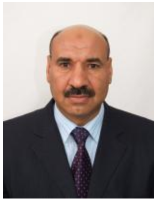

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